Axler 4A: Differentiation

I'm not so sure I like Roydens way of doing things here.

The issue with Royden is that his proof of Vitali & differentiation is a bit convoluted:

1) f is differentiable a.e. if f is monotone

2) $\int f' < f(b) - f(a)$ using Fatou.

3) BV fns. BV fns can be written as a difference of 2 fns.
So BV fns are a.e. differentiable

4) Absolute continuity: If f is absolutely continuous then

f is differentiable a.e. and $\int_a^b f' = f(b) - f(a)$

5) Finally if $f \in L^1$ $\int_a^x f = F(x)$ and $F'(x) = f(x)$ a.e.

(Lebesgue differentiation)

In contrast, Axler is a lot more efficient.
- Vitali is very clean, and focuses on the essential idea.
- Hardy-Littlewood is a beautiful idea.
- Lebesgue diff is almost trivial from this.
- The Lebesgue density theorem is rather nice.

Maybe its worth doing some bits of Royden, like the BV & absolute continuity part.

Vitali covering lemma

I - interval. $3I$ same center as I, but 3 times its length.

Lemma: Let $\{I_i\}_{i=1}^{n}$ bounded, open. Then $\exists$ a disjoint sublist $\{I_{k_j}\}$ st

$$I_1 \cup I_2 \cdots \cup I_n \subseteq \bigsqcup_j I_{k_j}$$

Pf: Let k_1 st $I_{k_1} = \max\{|I_1|, \dots |I_n|\}$

(longest interval). Then find I_{k_2} disjoint from I_{k_1} st I_{k_2} is maximal & so on

Suppose we stop at some point j & no disjoint intervals remain. $I_{k_j} = \max\{|I_r| \dots\}$

$$\overset{I_{k_q}}{(\ \underset{a}{|}\ (\cdot)\ \overset{b}{|}\ )}_{I_r}$$

Then $|I_r| \leq I_{k_q} \Rightarrow I_r \subseteq 3I_{k_q}$ (for sure)

Basically $|b-a| \leq |I_{k_j}|$

Thats it! Nuts.

Hardy littlewood fuction

$$h^*(b) = \sup_{t>0} \frac{1}{2t}\int_{b-t}^{b+t} h(s)\,ds$$

local average

Thats the maximal part

let

$h = 1_{[0,1]}$ Then $h^*(b) = \begin{cases} \frac{1}{2(1-b)} & b \leq 0 \\ 1 & 0 < b < 1 \\ \frac{1}{2b} & b \geq 1 \end{cases}$

Maximal inequality

$$m(\{ b : h^*(b) > c \}) \leq \frac{3}{c} \|h\|_{L^1}$$

Compare with Markov / Chebyshev. $m(b : |h(b)| > c) \leq \frac{1}{c} \int |h|$

Note that Chebyshev (1800s) was Markov's advisor.

let $F \subset \{ b : h^*(b) > c \}$ be any closed and bounded subset.

Enough to show $m(F) \leq \frac{3}{c} \|h\|_{L^1}$

$\forall\, b \in F,\ \exists\ t_b$ st $\frac{1}{2t_b} \int_{b-t_b}^{b+t_b} h > c$. Consider the collection of such intervals $(b-t_b, b+t_b)$.

These must cover F! By Heine-Borel, $\exists$ a finite subcover $I_{b_1}, \dots I_{b_n}$

and by Vitali there is a disjoint collection $I_{k_1} \dots I_{k_j}$ st $F \subset 3I_{k_1} \sqcup \dots \sqcup 3I_{k_j}$

That's it!

$$|F| \leq 3 \sum_{i=1}^{j} |I_{k_j}| \leq 3 \sum_{i=1}^{j} \int_{I_{k_j}} \frac{1}{c} |h| \leq \frac{3}{c} \int |h|$$ Amazing and efficient.

4B Lebesgue differentiation

Thm: If $f \in L^1$ (f is integrable)

$$\lim_{t \downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)| \doteq 0 \quad \text{for a.e } b \in \mathbb{R}.$$

Pf: If f continuous at b $\quad \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)| < \sup\{ |f(x) - f(b)| : |x-b| < t \}$

$< \epsilon$ for all t small.

$\Rightarrow$ APPROXIMATE f by continuous functions.

$\forall\ \delta > 0\ \exists\ h_k$ st $\|f - h_k\| < \dfrac{\delta}{k 2^k}$, h_k continuous.

Corollary of Luzin. (maybe HW, exam)

Let

$$B_k = \{ b : |f - h_k| \leq \tfrac{1}{k} \text{ and } (f - h_k)^* \leq \tfrac{1}{k} \}$$

$$m(\mathbb{R} \setminus B_k) \leq m\left(|f - h_k| > \tfrac{1}{k} \right) + m\left(|(f - h_k)^*| > \tfrac{1}{k} \right)$$

$$\leq \frac{\delta}{2^k} + \frac{3\delta}{2^k}$$

$$m\Big(\big(\underbrace{\textstyle\bigcap B_k}_{B}\big)^c \Big) = m\left(\bigcup (\mathbb{R} \setminus B_k) \right) \leq \sum \frac{4\delta}{2^k} < 4\delta$$

Then on B, $\quad \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)| \leq \frac{1}{2t} \int_{b-t}^{b+t} |f - h_k| + |h - h_k(b)| + |h_k(b) - f(b)|$

$$\leq \frac{1}{k} + \frac{1}{k} + \frac{1}{2t}\int_{b-t}^{b+t} |h - h_k(b)|$$

by condition on B_k

small by continuity for all t small enough (which could depend on k)

$$\Rightarrow \overline{\lim_{t \downarrow 0}} \frac{1}{2t}\int_{b-t}^{b+t} |f - f(b)| < \frac{3}{k} \quad \forall k.$$

$$\Rightarrow \lim_{t \downarrow 0} \frac{1}{2t}\int_{b-t}^{b+t} |f - f(b)| = 0 \quad \text{on } B$$

let $A = \left\{ b : \overline{\lim_{t\downarrow 0}} \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)| \neq 0 \right\}$

$A \subset \mathbb{R} \setminus B$ (since on B it is 0)

$\Rightarrow m(A) \leq m(\mathbb{R}\setminus B) < 4\delta$, but δ is arbitrary.

<u>Differentiation</u>: $g : I \longrightarrow \mathbb{R}$, $b \in I$, I nonempty & open.

$$g'(b) = \lim_{t \downarrow 0} \frac{g(b+t) - g(b)}{t}$$

FTC: let $f \in L^1$, and define

$$g(x) = \int_{-\infty}^{x} f.$$

Suppose f is continuous at b then $g'(b) = f(b)$

Note f has to be continuous, if not you can change the value at b alone:

$$\hat{f}(x) = \begin{cases} f(x) & x \neq b \\ a \end{cases} \qquad \text{where } a \neq f(b)$$

Then $g(x) = \int_{-\infty}^{x} \hat{f}$ and $g'(b) \neq \hat{f}(b)$

$$\left| \frac{g(b+t) - g(b)}{t} - f(b) \right| = \left| \frac{\int_b^{b+t} f}{t} - f(b) \right| = \left| \frac{1}{t} \int_t^{b+t} f - f(b) \right|$$

Choose t small st $|f(x) - f(b)| < \epsilon \quad \forall\ |x - t| < \delta$

$\leq \quad \epsilon$ and done.

Didn't really need anything for this. But we need this extra hypothesis of continuity at b and it doesn't really seem that important.

Lebesgue differentiation theorem v2

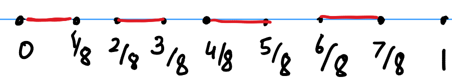
let $f \in L^1$ then g is $g'(b) = f(b)$ for a.e. b

Pf: Do the same calc. as before

$$\left| \frac{1}{t} \int_t^{b+t} f - f(b) \right| \longrightarrow 0$$ a.e. $b \in \mathbb{R}$ by previous.

Question: Does there exist $E \subset [0,1]$ st $m(E \cap [0,b]) = \frac{b}{2}$ $\forall\, b \in [0,1]$

Ex: $E = [0, 1/8] \cup [2/8, 3/8] \cup [4/8, 5/8] \cup [6/8, 7/8]$



Works for $2/8, 4/8, 6/8, 1$.

Thm: No

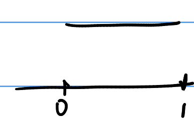
Pf: Suppose E exists; $g(b) = \int_{-\infty}^{b} 1_E = m(E \cap [0,b]) = \frac{b}{2}$

$\Rightarrow$ $g'(b) = \frac{1}{2}$ which contradicts $g'(b) = 1$ a.e. by Lebesgue diff.

Remark: Let $f \in L^1$. Then $f(b) = \lim_{t \downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} f$ "function is equal to its local average everywhere."

$$\left| \frac{1}{2t} \int_{b-t}^{b+t} f - f(b) \right| \le \left| \frac{1}{2t} \int_{b-t}^{b+t} f - f(b) \right| \longrightarrow 0 \quad b \text{ a.e.}$$

Density: What is the density of



$$d_E(b) = \begin{cases} 1/2 & b \in \{0,1\} \\ 1 & b \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

$$d_E(b) = \lim_{t \downarrow 0} \frac{m(E \cap (b-t, b+t))}{2t}$$

$$= \lim_{t \downarrow 0} \frac{1}{2t} \int_{b-t}^{b+t} \mathbb{1}_E$$

<u>Density thm</u>: let $E \subset \mathbb{R}$, then $d_E(b) = \begin{cases} 1 & \text{a.e. on } E \\ 0 & \text{a.e. on } \mathbb{R} \setminus E \end{cases}$

Pf: If $m(E) < \infty$ then theorem follows from Lebesgue differentiation.
since $\mathbb{1}_E \in L^1$

If $m(E) = +\infty$ For $k \in \mathbb{Z}^+$ $E_k = E \cap (-k, k)$

Then if $b \in (-k, k)$ $d_E(b) = d_{E_k}(b) = \begin{cases} 1 & \text{a.e. } E_k \text{ by previous} \\ 0 & \text{a.e. } \mathbb{R} \setminus E_k \text{ otherwise.} \end{cases}$

$= 1$ on $E_k \setminus F_k$ where $m(F_k) = 0$
$= 0$ on $E_k^c \setminus G_k$ $m(G_k) = 0$

G_k can only be $\{\pm b\}$ (You can take the bad set as $\hat{E}_k$ and set in the Lebesgue diff theorem
$F_k = E_k \cap \hat{E}_k$ and $G_k = (E_k^c \cap \hat{E}_k)$)

let $F = \bigcup_{k=1}^{\infty} F_k$, $G = \bigcup_{k=1}^{\infty} G_k$

$\forall x \in E \setminus F$ $d_E(x) = d_{E_k}(x) = 1$ $\forall k > |x|$ and $m(F) \leq \sum m(F_k) = 0$

and same for $E^c \setminus G$

Thm:

There is a bad Borel set.

$\exists\ E \subset \mathbb{R}$ st $\quad 0 < |E \cap I| < |I| \quad \forall$ bounded open interval I.

$$f(x) = \int_a^x \mathbb{1}_E < x - a \quad \forall\ x$$